Markov Chains

- Probability Distributions of Markov Chains (III.M.1)
- Long-run Behavior (III.M.2)
Example: At a university it is estimated that 60% of the commuter students drive to school and the other 40% take the bus. The university is upgrading the bus system and expects bus usage to increase in the next semester. It is projected that 20% of the current car drivers will switch to using the bus and 90% of the current bus users will continue to use the bus.

a) Make a tree diagram representing the bus and car usage.

b) What percent of students will use each transportation method at the beginning of the second semester, i.e., after 1 semester?
c) What percent of students will use each transportation method after 2 semesters?

d) After 10 semesters?
Markov Chains

- A *Markov chain* or *process* describes an experiment consisting of a finite number of stages.
  - The outcomes and associated probabilities at each stage depend only on the outcomes of the preceding stage.
- The outcome at any stage of the experiment in a Markov chain is called the *state* of the experiment.
Matrices for Markov Chains

- A *stochastic matrix* is a square matrix such that:
  - All entries are nonnegative.
  - The entries in each column sum to 1.

- A *transition matrix* $T$ is an $n \times n$ stochastic matrix associated with a Markov chain of $n$ states.
  - Entries represent conditional probabilities, i.e., the probabilities from the second stage of the associated tree.
  - $t_{ij} = P(X_i | X_j)$ where $X_i$ and $X_j$ are states.

- The initial state is stored as matrix $X_0$. 
Example continued: Use a transition matrix to find the percent of students using each transportation method after 2 semesters and after 3 semesters.

\[ T = \]

The initial state goes into a distribution matrix, a.k.a., initial-state vector.

\[ X_0 = \]

Multiply to find the next state which corresponds to 1 semester later:

\[ X_1 = T \cdot X_0 = \]
Example continued:

After 2 semesters:

\[ X_2 = T \cdot X_1 = \]

After 3 semesters:

\[ X_3 = T \cdot X_2 = \]

After 10 semesters:

\[ X_{10} = \]
Summary

- The previous example is a Markov process because to calculate a result, we needed to know only one prior state.
- The transition matrix $T$ is given by:
  \[
  T = \begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
  \end{bmatrix} = \begin{bmatrix}
  P(S1|S1) & P(S1|S2) \\
  P(S2|S1) & P(S2|S2)
  \end{bmatrix}
  \]
- The initial state is given by:
  \[
  X_0 = \begin{bmatrix}
  p_1 \\
  p_2
  \end{bmatrix}
  \]
- The probability distribution of the system, given by state $X_m$, after $m$ observations is $X_m = T^m X_0$. 
Example: A town has three restaurants that sell tacos. A study found that 75% of those who dine at Reyna’s Diner during a particular week will return to Reyna’s in the following week, but 15% will dine at Matamoros Taco House in the following week, and the rest will go to Senor Taco. Of those who dine at Matamoros Taco House in a particular week, 5% will then go to Reyna’s Diner, 15% will then go to Senor Taco in the next week, and the rest will return to Matamoros. Of those who dine at Senor Taco in a particular week, 10% will go to Reyna’s, 20% will go to Matamoros Taco House, and the rest will return to Senor Taco in the next week. This week, 25% of those craving tacos went to Reyna’s Diner, 33% went to Matamoros Taco House, and the rest went to Senor Taco. What percentage of customers will visit each of the three restaurants after 5 weeks?
Example: Recall the transition matrix $T$ and initial distribution from the bus and car example:

$$T = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \quad X_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

What is the distribution of the bus and car usage in later semesters?
Regular Markov Chains

- A *steady-state matrix* describes the long-term behavior of a regular Markov chain.
- A stochastic matrix $T$ is a regular Markov chain if the sequence $T, T^2, T^3, \ldots$ approaches a steady-state matrix with all strictly positive entries.
- The columns of the steady-state matrix for a regular Markov chain will all be equal to the steady-state distribution vector.
Regular Matrices

- A stochastic matrix is called \textit{regular} if some power of the matrix has all positive entries.

\textit{Example:} Which of these matrices are regular?

\[ T_A = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \quad T_B = \begin{bmatrix} 0.7 & 0 \\ 0.3 & 1 \end{bmatrix} \quad T_C = \begin{bmatrix} 0 & 0.6 \\ 1 & 0.4 \end{bmatrix} \]
Finding the Steady-State Distribution Vector

- For a regular matrix $T$, solve $TX = X$.
- Incorporate the condition that the sum of entries of $X$ is 1.
Example: Recall the transition matrix $T$ from the bus/car example:

$$
T = \begin{bmatrix}
0.9 & 0.2 \\
0.1 & 0.8
\end{bmatrix}
$$

Solve the system $TX = X$ to find the steady-state distribution vector.
Example: In a survey of those who listened to classical music (C) during one month, the next month 20% listened to classical and the rest listened to oldies (O). Of those who listened to oldies in one month, the next month 40% listened to classical and the rest listened to jazz (J). Of those who listened to jazz in one month, the next month 50% listened to classical, 35% listened to oldies, and 15% listened again to jazz. What is the long-term distribution of music listeners?