Uniform Sample Spaces and Probabilities

- If $E$ is any event of a uniform sample space, then

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{Number of Outcomes in } E}{\text{Number of Outcomes in } S}$$

- It sometimes requires counting techniques to determine $n(E)$ and $n(S)$.
Example: Assuming the probability of a boy being born is equal to that of a girl being born, find the probability that a family with four children (all born at different times) will have

a) No boys?

b) At least one boy?

c) Exactly two boys?
Example: A multiple choice exam has four questions, each with five possible answers. What is the probability of a monkey guessing exactly three questions correctly?
Example: You have a jar with 5 green, 3 red, and 7 blue marbles. If you select four marbles at random from the jar, what is the probability that you select

a) Exactly one red and three blue marbles?

b) At least two green marbles?
Example continued:

c)  All marbles of the same color?

d)  Exactly one red or exactly three blue marbles?
Example: A group of six people is selected at random. What is the probability that at least two of them have the same birthday?
Conditional Probability

- **Idea:** You have prior information regarding the occurrence of a particular event of an experiment. Hence, the probability of another event occurring might change, relative to this new information.

- The probability that event $B$ occurs, given that event $A$ has occurred is given by

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) \neq 0$$
Example: A poll was conducted among 150 students regarding smoking in public restaurants. The results of the poll are shown in the table:

<table>
<thead>
<tr>
<th></th>
<th>Favors Smoking</th>
<th>Opposes Smoking</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>45</td>
<td>35</td>
<td>80</td>
</tr>
<tr>
<td>Female</td>
<td>10</td>
<td>60</td>
<td>70</td>
</tr>
<tr>
<td>Total</td>
<td>55</td>
<td>95</td>
<td>150</td>
</tr>
</tbody>
</table>

a) What is the probability that a randomly selected male opposes smoking?

b) What is the probability that a randomly selected proponent of smoking is male?
Example: A pair of fair six-sided dice is rolled. What is the probability that a 2 is rolled, if it is known that the sum of the numbers rolled is four?
Example: The probability that a car will last 5 years or more is 0.95 and the probability that a car will last 10 years or more is 0.45. Given that a car has lasted for 5 years, what is the probability of it lasting 10 years or more?
Example: Let \( A \) and \( B \) be two events from a sample space, \( S \), where \( P(A) = 0.8 \), \( P(B^c) = 0.4 \), and \( P(A \cup B) = 0.9 \). Find

a) \( P(B \mid A) \)

b) \( P(A \mid B) \)
The Product Rule

- The probability that event $A$ and event $B$ both occur is given by

$$P(A \cap B) = P(A)P(B \mid A)$$

or by

$$P(A \cap B) = P(B)P(A \mid B)$$
Example: Let $A$ and $B$ be two events from a sample space, $S$, where $P(A^c) = 0.4$ and $P(B \mid A) = 0.5$. Find $P(A \cap B)$. 
Tree Diagrams

- Used to see relationships between different stages of an experiment with a finite number of stages.
- All branches leading from one “node” represent all the possible outcomes from that point in that stage of the experiment.
- Each branch, representing an outcome, can be labeled with the probability of that outcome occurring.
- Each branch stemming from a node after the initial node represents a conditional outcome, labeled by a conditional probability.
Example: You toss a fair coin, noting the side landing up, and then randomly select a letter from the word MATH.

a) Draw a tree diagram to represent this problem. Clearly label the probabilities on each branch.

b) What is the probability that an H is selected from MATH if the coin lands heads up?
c) What is the probability that the coin lands heads up and you draw an H?

d) What is the probability that an H is selected from the word MATH?
**Example:** In a particular subdivision of 85 homes, 70 of these have fertilized lawns. Of those homes where the lawn is fertilized, 10% have unhealthy patches of grass, whereas 95% of the homes without fertilized lawns have unhealthy patches of grass.

a) Draw a tree diagram to represent this problem. Clearly label the probabilities on each branch.
Example continued:

b) What is the probability that a randomly selected home in this subdivision with a fertilized lawn has no unhealthy patches of grass?

c) What is the probability that a randomly selected home in this subdivision has a fertilized lawn with unhealthy patches of grass?

d) What is the probability that a randomly selected home in this subdivision has unhealthy patches of grass?
Independent Events

- Two events are independent if the outcome of one event does not affect the probability of the outcome of the other.

- If $A$ and $B$ are independent events, then
  
  $$P(A \mid B) = P(A) \quad \text{and} \quad P(B \mid A) = P(B)$$

- Two events, $A$ and $B$, are independent if and only if
  
  $$P(A \cap B) = P(A)P(B)$$
Example: From the previous two examples, are

a) “Drawing an H from the word MATH” and “the coin lands on heads” independent events? Why or why not?

b) “Having a fertilized lawn” and “having unhealthy patches of grass” independent events? Why or why not?
Example: Kathryn, Kevin, and Mark each remember their mom’s birthday with probabilities 0.99, 0.98, and 0.975, respectively. If none of them talk to each other or anyone else knowing their mom’s birthday this year, prior to their mom’s birthday, what is the probability that

a) They all remember their mom’s birthday?

b) Exactly one of them remembers their mom’s birthday?
Bayes’ Theorem

**Idea**: You are computing the likelihood of an event occurring based upon information “after the fact”.

Let $S$ be a sample space partitioned into $n$ events, $A_1, A_2, \ldots, A_n$. Let $E$ be any event of $S$ where $P(E) \neq 0$. The probability of the event $A_i$ ($i=1,2,\ldots,n$), given the event, $E$, is

$$P(A_i \mid E) = \frac{P(A_i)P(E \mid A_i)}{P(E)}$$

$$= \frac{P(A_i)P(E \mid A_i)}{P(A_1)P(E \mid A_1) + P(A_2)P(E \mid A_2) + \cdots + P(A_n)P(E \mid A_n)}$$
Special Case of Bayes’ Theorem

Find a simple formula for $P(A \mid E)$, given the tree diagram below:
Example:

a) Fill in the missing probabilities on the given tree.

\[ P(E | A) = \]

b) \( P(A \cap E) = \)

c) \( P(A \cap E) = \)

d) \( P(E) = \)

e) \( P(A | E) = \)

f) \( P(C | E) = \)
Example: A particular shirt is sold in only three stores. Records show Store A places the shirt on sale 10% of the time, Store B only 5% of the time and Store C 20% of the time. It is decided by the distributor that Store A will sell 20% of the shirts made, Store B will sell 65% of the shirts made and Store C will sell the rest. If a customer buys the shirt on sale, from which store did she most likely buy the shirt?
Example: Two cards are drawn in succession, without replacement, from a well-shuffled standard 52-card deck.

a) What is the probability that the first card is a heart, given that the second card is a diamond?
Example continued:

b) What is the probability that the first card is a red card, given that the second card is a heart?