Probability

- Experiments, Sample Spaces, and Events (II.B.1a,1c)
- Definition of Probability (II.B.1b,1d,2,3a)
- Rules of Probability (II.B.3)
- Counting Techniques Used in Probability (II.B.4)
- Conditional Probability and Independent Events (II.B.4a,4b,4c)
- Bayes’ Theorem (II.B.4d)
Experiments

- Activities with observable results.
- Results are known as *outcomes*.

*Examples:*

a) Tossing a coin and noting the side landing uppermost.

b) Rolling a six-sided die and noting the number showing uppermost.

c) Choosing a piece of candy from a bowl and noting the type of candy selected.
Sample Points, Sample Spaces and Events

- A sample point is an outcome of an experiment.
  - It is an element of a set.

- The sample space ($S$) is the set consisting of all possible sample points of an experiment.
  - It is a universal set.

- An event is a collection of sample points from an experiment.
  - It is a subset of the sample space.
  - An event is said to occur whenever the event contains the observed outcome.
Types of Sample Spaces

- **Finite Sample Spaces**
  - The experiment has a finite number of possible sample points (or outcomes).
  - All sample points can explicitly be listed.

- **Non-Finite Sample Spaces**
  - The experiment has an infinite number of possible sample points.
Example: You toss a fair coin and note which side lands uppermost.

a) Describe the sample space associated with this experiment.

b) What are the sample points?

c) What are the events?
Mutually Exclusive Events

- Events that cannot occur at the same time.
- They do not share any common sample points.
- Two events, $A$ and $B$, are *mutually exclusive* if $A \cap B = \emptyset$.

*Example*: In the previous coin example, are any non-empty events mutually exclusive?
Example: An experiment consists of rolling a fair six-sided die (noting the number landing up) and tossing a fair coin (noting the side landing up).

a) Describe the sample space associated with this experiment.

b) Describe the event, $E$, that your coin lands heads up.

c) Describe the event, $F$, that your die lands on an even number.

d) Are $E$ and $F$ mutually exclusive? Why or why not?
Example: Two fair six-sided dice are cast (one red and one blue) and the numbers falling uppermost are observed.

a) Describe the sample space associated with this experiment.

b) Write the event, $E$, that a sum of 6 is rolled.

c) Write the event, $F$, that a product of 6 is rolled.

d) Are $E$ and $F$ mutually exclusive? Why or why not?
Example: You go to a football game and observe the length of time you stand during the game (in minutes).

a) Describe the sample space for this experiment.

b) Describe the event, $E$, that you stand between 100 and 200 minutes.

c) Describe the event, $F$, that you stand for no more than 60 minutes.
Empirical Probability

- The relative frequency of an event $E$ is $\frac{m}{n}$ if in $n$ trials of an experiment, $E$ occurs $m$ times.

- As the number of trials in an experiment becomes larger, if the relative frequency of event $E$ approaches a certain value, we call this value $P(E)$, the empirical probability of $E$.

- $P(E)$ is a measure of the proportion of total trials in which an event occurs in the long run.

- The probability of an event occurring is ALWAYS a number that lies between 0 and 1, inclusive.
Example: Two cities on a planet in another solar system were surveyed for life and the following information was gathered.

<table>
<thead>
<tr>
<th></th>
<th>Dogs</th>
<th>Cats</th>
<th>Birds</th>
<th>Fish</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>City A</td>
<td>12</td>
<td>46</td>
<td>24</td>
<td>88</td>
<td>170</td>
</tr>
<tr>
<td>City B</td>
<td>60</td>
<td>15</td>
<td>7</td>
<td>28</td>
<td>110</td>
</tr>
<tr>
<td>Total</td>
<td>72</td>
<td>61</td>
<td>31</td>
<td>116</td>
<td>280</td>
</tr>
</tbody>
</table>

If one animal is selected at random, what is the probability that

a) A bird is selected?

b) An animal from City A is selected?
Simple Events

- Events that contain exactly one sample point.
- Together with their probabilities they form a probability distribution for the experiment.

Properties:
- Mutually exclusive.
- $0 \leq P(\{s_i\}) \leq 1$ for each simple event, $\{s_i\}$.
- $P(\{s_i\} \cup \{s_j\}) = P(\{s_i\}) + P(\{s_j\})$ (where $i \neq j$)
- Sum of their probabilities is one:

$$P(\{s_1\}) + P(\{s_2\}) + \cdots + P(\{s_n\}) = 1$$
Uniform Sample Spaces

- Sample spaces in which outcomes are equally likely.
- In a uniform sample space with \( n \) sample points, the probability of each outcome, \( s_1, s_2, \ldots, s_n \), is \( \frac{1}{n} \).
Example: A fair die is rolled and the outcome is recorded.

a) What is the sample space associated with this experiment?

b) List all of the simple events.

c) What is the probability of each simple event?

d) Find the probability distribution for this experiment.

e) What is the probability a two or a four is rolled?
**Example:** A group of people was asked to name their favorite color. All of their responses are summarized in the frequency distribution below.

<table>
<thead>
<tr>
<th>Color</th>
<th>Blue</th>
<th>Yellow</th>
<th>Purple</th>
<th>Red</th>
<th>Green</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>45</td>
<td>5</td>
<td>12</td>
<td>10</td>
<td>8</td>
</tr>
</tbody>
</table>

a) Find the probability distribution for this experiment.

b) Is this a uniform sample space? Why or why not?

c) What is the probability that a randomly selected person from this group named red or blue as his/her favorite color?

d) What is the probability that a randomly selected person from this group named burnt orange as his/her favorite color?
Example: A pair of fair dice is cast.

a) What is the probability that the two dice show different numbers?
b) What is the probability that a four is rolled?
c) What is the probability that the sum of the two numbers is 2? 8? 13?
Example: An experiment consists of picking one letter out of the word OCTOBER.

a) Write a uniform sample space associated with this experiment.

b) Write a non-uniform sample space associated with this experiment.
Rules of Probability

Let $S$ be a sample space of an experiment and suppose $E$ and $F$ are events of the experiment. Then,

- $0 \leq P(E) \leq 1$ for any event $E$
- $P(S) = 1$
- If $E$ and $F$ are mutually exclusive, then
  \[ P(E \cup F) = P(E) + P(F) \]
- If $E$ and $F$ are any two events, then
  \[ P(E \cup F) = P(E) + P(F) - P(E \cap F) \]
- $P(E^C) = 1 - P(E)$
Example: The grades on an exam were distributed as shown. If a student is selected at random from this class, what is the probability that his or her score on the exam is

<table>
<thead>
<tr>
<th>Grade, $x$</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \geq 90$</td>
<td>11</td>
</tr>
<tr>
<td>$80 \leq x &lt; 90$</td>
<td>30</td>
</tr>
<tr>
<td>$70 \leq x &lt; 80$</td>
<td>42</td>
</tr>
<tr>
<td>$60 \leq x &lt; 70$</td>
<td>57</td>
</tr>
<tr>
<td>$x &lt; 60$</td>
<td>10</td>
</tr>
</tbody>
</table>

a) At least an 80?

b) A passing grade?

c) At least a 70, but less than a 90?
Example: Let $E$ and $F$ be two events of an experiment. Suppose $P(E)=0.6$, $P(F)=0.4$ and $P(E \cap F) = 0.2$. Find the following:

a) $P(E \cup F)$

b) $P(E^C)$

c) $P(E^C \cap F)$

d) $P(E^C \cup F^C)$
Example: Let \( S = \{s_1, s_2, s_3, s_4\} \) be the sample space for an experiment with the following distribution:

<table>
<thead>
<tr>
<th>Outcome</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>( s_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{2}{4} )</td>
<td>( \frac{1}{4} )</td>
<td></td>
</tr>
</tbody>
</table>

a) Fill in the missing probability in the distribution table.
b) Is the sample space uniform? Why or why not?
c) If \( A = \{s_1, s_3\} \) and \( B = \{s_1, s_2, s_4\} \), find the following probabilities:

\[
P(A) \\
P(B) \\
P(A \cap B) \\
P(A \cup B) \\
P(A^C \cap B) \\
P(A^C \cup B^C)
\]
Example: An experiment consists of selecting one card at random from a well-shuffled standard 52-card deck. Find the probability that the card drawn is

a) A queen of spades.

b) A queen or a spade.

c) A spade or a heart.

d) Not a two.
Example: A survey of 500 students found that 350 liked to watch television, 200 liked to listen to the radio and 100 liked to watch television and listen to the radio. What is the probability that a student selected at random from this group

a) Liked to watch TV and listen to the radio?

b) Liked at least one of the two mentioned activities?

c) Liked exactly one of the two activities?

d) Liked neither of the two activities?
Example: A poll was conducted among 150 students regarding smoking in public restaurants. The results of the poll are shown in the table:

<table>
<thead>
<tr>
<th></th>
<th>Favors Smoking</th>
<th>Opposes Smoking</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>45</td>
<td>35</td>
<td>80</td>
</tr>
<tr>
<td>Female</td>
<td>10</td>
<td>60</td>
<td>70</td>
</tr>
<tr>
<td>Total</td>
<td>55</td>
<td>95</td>
<td>150</td>
</tr>
</tbody>
</table>

If one of the students in this poll is selected at random, what is the probability that
a) The student is a male?
b) The student favors smoking in restaurants?
c) The student is a male and favors smoking in restaurants?
d) The student is a male or favors smoking in restaurants?
e) The student is not a male or does not favor smoking in restaurants?