Mathematics of Finance

- Compound Interest (III.F.1)
- Annuities and Sinking Funds (III.F.2)
Simple Interest

- *Simple interest* is interest earned or paid on the original principal only.

- **Formula:** \( I = P \cdot r \cdot t \)
  - \( I \) represents
  - \( P \) represents
  - \( r \) represents
  - \( t \) represents

- **Total earned:** \( A = P + I = P(1 + rt) \)
Example: $100 is invested in an account paying 5% simple interest per year.

- How much is in the account after 10 years?

- How much interest is earned?
Example: If you borrow $500 for 18 months at a rate of 13% simple interest per year, how much will you owe after 18 months?
Compound Interest

- *Compound interest* is interest earned or paid on both the principal and interest.
Example: $100 is invested in an account paying 5% interest per year compounded annually. How much is in the account after 10 years?

end of year 1:

end of year 2:

end of year 3:

...

end of year 10:
Different Compounding Periods Per Year

<table>
<thead>
<tr>
<th>Compounding</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>annually</td>
<td>1</td>
</tr>
<tr>
<td>semi-annually</td>
<td>2</td>
</tr>
<tr>
<td>quarterly</td>
<td>4</td>
</tr>
<tr>
<td>monthly</td>
<td>12</td>
</tr>
<tr>
<td>weekly</td>
<td>52</td>
</tr>
<tr>
<td>daily</td>
<td>365</td>
</tr>
</tbody>
</table>
- \( N = \)
- \( I\% = \)
- \( PV = \)
- \( PMT = \)
- \( FV = \)
- \( P/Y = \)
- \( C/Y = \)
- \( PMT: END\quad BEGIN\)
Example: $100 is invested in an account paying 5% interest per year compounded annually. How much is in the account after 10 years?

- **N** = 
- **I%** = 
- **PV** = 
- **PMT** = 
- **FV** = 
- **P/Y** = 
- **C/Y** = 
- **PMT: END** \ **BEGIN**
Example: If an account paying 7% interest per year compounded annually has a balance of $2000 after 12 years, how much was initially invested in the account?

- N =
- I% =
- PV =
- PMT =
- FV =
- P/Y =
- C/Y =
- PMT: END  BEGIN
Example: If $100 is invested in an account paying 7.5% annual interest compounded monthly, how much will be in the account after 10 years?

- N =
- I% =
- PV =
- PMT =
- FV =
- P/Y =
- C/Y =
- PMT: END  BEGIN
Example: Find the amount at the end of 5 years if you invest $3000 at 5.25% annual interest compounded quarterly.

- N =
- I% =
- PV =
- PMT =
- FV =
- P/Y =
- C/Y =
- PMT: END   BEGIN
Example: How much money should you deposit in a bank account paying 6.1% per year compounded weekly so that at the end of 4 years you will have $8000?

- N =
- I% =
- PV =
- PMT =
- FV =
- P/Y =
- C/Y =
- PMT: END  BEGIN
Example: You deposit $3500 in a savings account paying 4.4% per year compounded semi-annually. After how many years will you have $5900 in the account?

- N =
- I% =
- PV =
- PMT =
- FV =
- P/Y =
- C/Y =
- PMT: END BEGIN
Let’s revisit a previous problem:

$100 is invested in an account paying 5% interest compounded annually. How much is in the account after 10 years?

How much would be in the account if the interest was compounded

- Weekly?
- Daily?
- Every hour of every day?
Continuous Compound Interest

- Formula: $A = Pe^{rt}$
  - $A$ is the accumulated amount at the end of $t$ years
  - $P$ is the principal
  - $e = 2.71828...$
  - $r$ is annual interest rate as a decimal compounded continuously
  - $t$ is time in years
Example: Find the amount at the end of 5 years, if you invest $3000 at 5.25% interest compounded continuously.
Effective Rate of Interest

- The effective rate is the simple interest rate that would produce the same amount of accumulated interest in one year.
- Used as a basis for comparing interest rates.
- Equivalent to Annual Percentage Yield (APY)
- When paying interest we want the lowest effective rate, and when earning interest we want the highest effective rate.
Example: What is the effective interest rate for an account paying 8% compounded monthly?
Example: A store credit card offers three choices of interest rates. The options are 18.1% annual interest compounded daily, 18.2% annual interest compounded monthly, and 18.5% annual interest compounded quarterly. Which option is the best choice for a person using this credit card?
Example continued: Assume you carry a balance of $1200 on this credit card and make no payments. What is the balance after one year? Check your answer by using your choice of options and the corresponding effective yield.

\[
\begin{align*}
N &= \quad I = \quad PV = \\
PMT &= \quad FV = \quad P/Y = C/Y = \\
\text{Effective Yield}
\end{align*}
\]
Effective Yield for Continuously Compounded Interest

The effective yield of an account earning continuously compounded interest is given by

$$r_{\text{eff}} = e^r - 1$$

where

- $e = 2.71828...$
- $r$ is annual interest rate as a decimal
Example: What is the effective interest rate for an account paying 8% compounded continuously?
Example: You are planning a trip to Florida in 2 years. You want $2000 available. You find an investment paying 10% compounded quarterly. How much do you need to invest now to have the money ready in 2 years?

\[
\begin{array}{ccc}
N = & I = & PV = \\
PMT = & FV = & P/Y = C/Y = \\
\end{array}
\]

Instead, you could save up for the trip by making regular quarterly payments into an account paying 10% interest compounded quarterly. What payments would you be making?

\[
\begin{array}{ccc}
N = & I = & PV = \\
PMT = & FV = & P/Y = C/Y = \\
\end{array}
\]

How much do you save by having the money available at the start?
Annuities

- An annuity is an account into which regular payments are made.
  - Examples include monthly mortgages and deposits into savings accounts.
- An annuity that is certain and simple has the following properties:
  - The payments are made at fixed time intervals
  - The periodic payments are of equal size
  - The payments are made at the end of the interval
  - The interest is paid at the end of the interval
**Example:** A couple pays $800 at the end of each 6 months for 7 years into an annuity paying 11.3% per year, compounded semi-annually. How much will they have saved at the end of seven years?

\[
\begin{align*}
N &= \quad I = \quad PV = \\
PMT &= \quad FV = \quad P/Y = C/Y = \\
\end{align*}
\]

How much interest will they have earned?
Example: Being savvy about retirement planning, at age 18 you start putting $250 monthly into an account that earns 8.75% interest per year compounded monthly. How much money will you have saved for retirement 50 years later?

\[ N = \quad I = \quad PV = \]
\[ PMT = \quad FV = \quad P/Y = C/Y = \]

How much will you have earned in interest?
Example: Your parents deposited $500 per year into a college fund paying 7% annual interest compounded annually starting when you were born. How much is available when you are 18?

N = \hspace{1cm} I = \hspace{1cm} PV =

PMT = \hspace{1cm} FV = \hspace{1cm} P/Y = C/Y =

How much interest is earned?
Example: You deposit $2000 per year into a retirement fund. If the money is deposited once per year in an account paying 8.2% annual interest compounded annually, how much is in the account after 10, 20, 30 and 40 years?

N = \hspace{1cm} I = \hspace{1cm} PV =
PMT = \hspace{1cm} FV = \hspace{1cm} P/Y = C/Y =

After 10 years,
After 20 years,
After 30 years,
After 40 years,
Example: You purchase a car for no money down and payments of $299 a month for 60 months with interest of 12% annual interest charged on the unpaid balance every month. What was the cash price of the car?

\[ \begin{align*}
N & = \\
I & = \\
PV & = \\
PMT & = \\
FV & = \\
P/Y & = C/Y =
\end{align*} \]

How much did you pay in interest?
Example continued: How do the payments and total amount of interest change with a 4-year (48-payment) loan?

\[ \text{N = } \quad \text{I = } \quad \text{PV = } \]
\[ \text{PMT = } \quad \text{FV = } \quad \text{P/Y = C/Y = } \]
Example continued: How is it we paid so much interest?

- At the end of the 1\textsuperscript{st} period, we owe interest on the outstanding balance of $13441.56

  - Monthly interest rate is \(12 \times \frac{\text{year}}{12 \text{ months}} = 1 \times \frac{\text{month}}{\text{year}}\)

  - Interest owed = 

  - Principal paid =

  - So now we owe:
Example continued: How is it we paid so much interest?

- At the end of the 2\textsuperscript{nd} period, we owe interest on the outstanding balance of $\_\_\_\_\_\_.

  - Interest owed =

  - Principal paid =

  - So now we owe:

- And we own this much of the car:
Equity

- *Equity* is how much of the item belongs to you and not the bank.
- Equity = Value of item – Amt owed the bank
- In the previous example, the equity in the car is $330.81 at the end of the second period.
Amortization and Sinking Funds

- To *amortize* is to liquidate a debt by making installment payments.
- A *sinking fund* is a fund set up for a specific purpose at some future date.
- The equity in our car example can be summarized in an *amortization table*. 
Example: You buy a $120,000 house. You make a $20,000 down payment and finance the remainder at 7.5% annual interest compounded monthly on the outstanding balance for 30 years.

a) What are the monthly payments?

N = I = PV =
PMT = FV = P/Y = C/Y =

b) How much of the first month’s payment is principal and how much is interest?
c) How much interest is paid over the life of the loan?

d) What is the equity after 1 year? 5 years? 15 years?
Example: Eight years ago the Wizard of Oz secured a bank loan of $8,000,000 to pay for a new palace. The term of the mortgage was 30 years with an interest rate of 6% per year compounded monthly on the unpaid balance. Because the interest rate dropped to 4.5% per year compounded monthly, the Wizard is thinking of refinancing his palace with a 20-year mortgage.

a) What is his current monthly mortgage payment?

N = 
I = 
PV = 
PMT = 
FV = 
P/Y = C/Y =

b) What is the current outstanding principal?

N = 
I = 
PV = 
PMT = 
FV = 
P/Y = C/Y =
c) What is the equity in the palace?

d) If he decides to refinance, what would the monthly mortgage payment be?

\[
\begin{align*}
N &= \quad I &= \quad PV &= \\
\text{PMT} &= \quad FV &= \quad P/Y = C/Y = 
\end{align*}
\]

e) Should the Wizard refinance, assuming no refinancing charges? Why or why not?