Linear Programming

- Graphing Systems of Linear Inequalities in Two Variables (I.C.1)
- Linear Programming (I.C.2)
Graphing a Linear Inequality

- Recall that the equation of a line can be expressed as:
  \[ ax + by + c = 0 \]

- A linear inequality can similarly be expressed as:
  \[ ax + by + c < 0 \]
  \[ ax + by + c \leq 0 \]
  \[ ax + by + c > 0 \]
  \[ ax + by + c \geq 0 \]
The line resulting from the corresponding equation is called the *boundary line*. The boundary line divides the plane into two halves.

One side of the line is the *solution set* of the inequality.

The inclusion of the line in the solution region depends on the type of inequality:
- If the inequality uses \( \geq \) or \( \leq \), draw the boundary line as solid.
- If the inequality uses \( > \) or \( < \), draw the boundary line as dashed.
Example: Graph $2x - 3y - 12 < 0$. 
Graphing a System of Linear Inequalities

- If there are two or more inequalities, i.e. a system, the solution set, called $S$, will be the region where they are true at the same time.

- Bounded vs Unbounded
  - The solution set of a system of linear inequalities is *bounded* if you can draw a circle around it.
  - Otherwise, it is *unbounded*.
Example: Sketch the solution set $S$ for the following system. Determine whether $S$ is bounded or unbounded.

\[ x + y \leq 8 \]
\[ y - 3x \geq 0 \]
Example: Sketch the solution set $S$ for the following system. Determine whether $S$ is bounded or unbounded.

\[
\frac{1}{2}x + y \leq 3 \\
-2 \leq x < 2 \\
y \geq 0
\]
Linear Programming

- A linear programming problem has two parts:
  - A *linear objective* function to be maximized or minimized
  - Constraints to which the objective function is subject
    - The solution set of the constraints is the *feasible region* of the linear programming problem.
Example: A company sends all new employees through a training program consisting of both classroom and on-the-job training in various divisions of the company. For classroom training, employees earn 100 credits per hour in research and development, 40 credits per hour in production, and 20 credits per hour in marketing. For on-the-job training, employees earn 10 credits per hour in research and development, 80 credits per hour in production, and 15 credits per hour in marketing. It costs the company $60 per hour for classroom training and $45 per hour for on-the-job training. Each new employee must earn at least 260 credits in research and development, 320 credits in production, and 120 credits in marketing. Set up the linear programming problem that can be used to determine the number of hours of each type of training that should be required in order to minimize the total cost.

Identify the variables.

Write the objective function.

Write the constraints.
Example: A finance company has $120 million to invest in stocks T and S. Since stock T is a riskier investment, management has stipulated that the total amount invested in stock S be at least five times more than the amount invested in stock T. Stock T is expected to return an average of 20% and stock S an average of 12%. Determine the total amount that should be invested in each stock in order to maximize returns.
Corner Point Theorem

- If a linear programming problem has a solution, then it must occur at a vertex, or corner point, of the feasible region $S$ associated with the problem.

- If the objective function $P$ is optimized at two adjacent vertices of $S$, then it is optimized at every point on the line segment joining these vertices.
  - There are infinitely many solutions to the problem.
Example continued: Use the corner points and objective function to identify the solution.
Max/Min Theorem

- Suppose we are given a linear programming problem with a feasible region $S$ and an objective function $P = ax + by$.
  
  - If $S$ is bounded, then $P$ has both a maximum and a minimum value on $S$. 
Max/Min Theorem Cont.

- If $S$ is unbounded and both $a$ and $b$ are nonnegative, then $P$ has a minimum value on $S$ provided that the constraints defining $S$ include the inequalities $x \geq 0$ and $y \geq 0$.

- If $S$ is the empty set, then the linear programming problem has no solution; that is, $P$ has neither a maximum nor a minimum value.
Method of Corners

1. Graph the feasible region $S$.

2. Find the exact coordinates of all corner points (vertices) of the feasible region.

3. Evaluate the objective function at each corner point.

4. Find the corner point that renders the objective function a maximum (or minimum).

   - If there is only one such corner point, then this corner point is a unique solution to the problem.
   - If the objective function is maximized (or minimized) at two adjacent corner points of $S$, there are infinitely many optimal solutions given by the points on the line segment determined by these two corners.
Example: Objective: Maximize $P = 6x + 4y$
subject to: $3x + 8y \leq 96$
$3x + 2y \leq 42$
$0 \leq x \leq 10$
$y \geq 0$
Example: Objective: Minimize $P = x + 4y$
subject to: $-4x + y \leq 2$
$2x - y \leq 1$
$x \geq 0$
y \geq 0
Example: Find the maximum and minimum of $P = 5x + 6y$ subject to:

- $5x + 7y \geq 35$
- $3x + 4y \leq 12$
- $x \geq 0$
- $y \geq 0$
Example: A frozen lemonade stand has 5 cups of lemon juice, 8 cups of sugar and 16 cups of water available to make both sweet and tangy frozen lemonades for tomorrow. A sweet frozen lemonade uses 0.25 cups of lemon juice, 0.5 cups of sugar and 0.5 cups of water. The tangy lemonade uses 0.25 cups of lemon juice, 0.25 cups of sugar and 1 cup of water. If the price of a sweet lemonade is $1.25 and a tangy lemonade is $1.00, how many of each type of frozen lemonade should be made to maximize revenue? Is anything leftover?