Section M.1: Markov Processes

**Definition:** A Markov process or Markov chain is a sequence of experiments (trials), performed at regular time intervals, where each trial has the same set of outcomes. These outcomes are called states. The outcome of the current experiment is referred to as the current state of the process. The probability of moving from one state to another state on the next trial depends only on the present state and not on past history.

A transition diagram shows the states of the Markov process and the processes of going from one state to another.

\[
\begin{align*}
\text{Prob}(A \mid A) &= 0.7 & \text{Prob}(A \mid B) &= 0.4 \\
\text{Prob}(B \mid A) &= 0.3 & \text{Prob}(B \mid B) &= 0.6
\end{align*}
\]
**Definition:** A transition matrix (or stochastic matrix) is a square matrix satisfying the following.

1. All entries are between 0 and 1 (inclusive).
2. The sum of the entries in the columns is 1.

Example: Is $T$ a stochastic matrix? If so, draw the transition diagram.

\[
T = \begin{bmatrix}
  A & B & C \\
  0.3 & 0.6 & 0.1 \\
  0.7 & 0.1 & 0.7 \\
  0 & 0.3 & 0.2 \\
\end{bmatrix}
\]
Example: The university is upgrading the bus system and expects bus usage to increase in the next 6 months. It is projected that 30% of the current car drivers will switch to using the bus and 90% of the current bus users will continue to use the bus. The university estimates that 60% of the commuter students drive to school and the other 40% take the bus. What will the bus and car usage look like in 6 months? Assuming that the trend continues, what will the bus and car usage look like in 12 months? in 36 months?

\[
p(C) = .6(1.7) + .4(.1) = .46
\]

\[
p(B) = .6(1.3) + .4(.9) = .54
\]

\[
\begin{bmatrix}
.7 & .1 \\
.3 & .1 \\
\end{bmatrix}
\begin{bmatrix}
.4 \\
.9 \\
\end{bmatrix}
= \begin{bmatrix}
.46 \\
.54 \\
\end{bmatrix}
\]

\[
T = \begin{bmatrix}
.7 & .1 \\
.3 & .9 \\
\end{bmatrix}
\]

\[
X_0 = \begin{bmatrix}
.4 \\
.4 \\
\end{bmatrix}
\]

\[
X_1 = \begin{bmatrix}
.46 \\
.54 \\
\end{bmatrix}
\]

initial state (distribution)
\[ X_1 = T X_0 = \begin{bmatrix} 0.46 \\ 0.54 \end{bmatrix} \]

\[ X_2 = T X_1 = \begin{bmatrix} 0.376 \\ 0.624 \end{bmatrix} = T (T X_0) = T^2 X_0 \]

\[ X_3 = T X_2 = T (T^2 X_0) = T^3 X_0 \]

\[ X_4 = T X_3 = T^4 X_0 \]

\[ X_5 = T X_4 = T^5 X_0 \]

\[ X_6 = T X_5 = T^6 X_0 = C \begin{bmatrix} 0.2663 \\ 0.7337 \end{bmatrix} \]
If we have $X_2 + \mathbb{T}$ find $X_5$.

\[ X_5 = T^5 X_0 \]

\[ X_5 = T^3 X_2 \]
Let $T$ be the transition matrix of a Markov process and $X_0$ be the initial state. Then the $m^{th}$ state is found by

$$X_m = T^m X_0$$

Each state, $X_m$, is also called a distribution state (vector).

Example: The transition matrix $T$ is for the transition diagram in first example. Explain the meanings for $T^2$ and $T^4$.

$$T = \begin{bmatrix} A & B \\ 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \quad T^2 = \begin{bmatrix} A & B \\ 0.61 & 0.52 \\ 0.39 & 0.48 \end{bmatrix} \quad T^4 = \begin{bmatrix} A & B \\ 0.5749 & 0.5668 \\ 0.4251 & 0.4332 \end{bmatrix}$$

If you start in A, the prob you end up in B is 0.3 in 1 step.

If you start in B, the prob you end up in A in 2 steps is 0.5008
Example: If you know $X_2$ and $T$, give the computation to compute $X_5$.

$$X_5 = \Gamma^3 X_2$$
Example A town has three pizza places: A, B, and C. Consider the experiment of customers ordering pizza weekly (most likely a college student since it repeated many times). This Markov process can be modeled by the transition diagram. Assume at the start of the process 30% of the customers ordered from place A, 20% of the customers ordered from place C and the rest from B.

A) Find the distribution vectors after the 3rd and 8th week.

B) If a customer started in B, what is the probability they would be in A after 3 weeks?

\[ X_0 = \begin{bmatrix} .3 \\ .5 \\ .2 \end{bmatrix} \]

\[ T = \begin{bmatrix} .5 & .4 & .25 \\ .3 & .2 & .5 \\ .2 & .4 & .25 \end{bmatrix} \]

\[ X_3 = T^3 X_0 = \begin{bmatrix} .3983 \\ .3214 \\ .2803 \end{bmatrix} \]

\[ X_8 = T^8 X_0 = \begin{bmatrix} .3980 \\ .3234 \\ .2796 \end{bmatrix} \]