Section M.3: Absorbing Markov Processes

**Definition:** A state of a Markov process is called **absorbing** if once in that state there is no chance of leaving that state.

Example: Does the Markov process represented by this transition matrix have an absorbing state? If yes, give which states are absorbing.

\[
T = \begin{bmatrix}
A & B & C \\
0.1 & 0.3 & 0 \\
0.35 & 0.4 & 1 \\
0.55 & 0.3 & 0 \\
\end{bmatrix}
\]
Example: Does the Markov processes represented by these transition diagrams have an absorbing state? If yes, give which states are absorbing.

Yes: C and D

Yes: A
Example: Find the absorbing state(s), if any, of the transition matrix. Find the limiting matrix.

\[
T = \begin{bmatrix}
A & B & C \\
1 & 0.05 & 0 \\
0 & 0.45 & 0.5 \\
0 & 0.5 & 0.5 \\
\end{bmatrix}
\]

\[A \text{ is absorbing}\]

\[
L = B \begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]
Definition: An **absorbing stochastic matrix**, absorbing transition matrix, is a stochastic matrix in which

1) there is at least one absorbing state

2) from any state it is possible to get to at least one absorbing state, either directly or through one or more intermediate states.

A Markov process with an absorbing stochastic matrix is said to be an absorbing Markov process.

Example: Does this transition diagram represent an absorbing Markov process?

No.

C and D can not get to A.
Example: Does the transition matrix represent an absorbing Markov process?

\[
T = \begin{bmatrix}
A & B & C & D \\
1 & 0 & 0 & 0 \\
0 & 0.8 & 0 & 0.6 \\
0 & 0 & 1 & 0 \\
0 & 0.2 & 0 & 0.4 \\
\end{bmatrix}
\]

\[\text{No}\]
Definition: An absorbing transition matrix is said to be in standard form when the absorbing states are listed before the nonabsorbing states.

\[
\begin{bmatrix}
-1 & \text{absorb} \\
\text{nonabsorb}
\end{bmatrix}
\]

Example: Convert the absorbing transition matrix to standard form.

\[
T = \begin{pmatrix}
\begin{array}{cccc}
A & C & B & D \\
1 & 0 & .1 & .2 \\
0 & 1 & .3 & .15 \\
0 & 0 & .4 & .25 \\
0 & 0 & .2 & .4
\end{array}
\end{pmatrix}
\]
Example: Give the transition matrix in standard form for this transition diagram.

![Transition Diagram]

\[
T = D \begin{bmatrix}
 0 & 0 & 0.15 & 0.15 \\
 0 & 1 & 0.3 & 0.15 \\
 0 & 0 & 0.25 & 0.15 \\
 0 & 0 & 0 & 0.2 \\
\end{bmatrix}
\]

\[\begin{bmatrix}
 0 & 1 & 0.15 & 0.15 \\
 0 & 0 & 0.15 & 0.25 \\
 0 & 0 & 0.15 & 0.2 \\
 0 & 0 & 0.25 & 0.15 \\
\end{bmatrix} \]
Theorem: (Part 1) If \( T \) the transition matrix of an absorbing Markov process is in standard form then the limiting matrix can be found by the following calculation. Note: This limiting matrix is sometimes called the stable matrix.

\[
T = \begin{bmatrix}
I & A \\
0 & B
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
I & A(I - B)^{-1} \\
0 & 0
\end{bmatrix}
\]

\[A^*(I - B)^{-1}\]
Example: Find the limiting matrix for this transition matrix.

\[ T = \begin{bmatrix} A & C & B & D \\ A & 1 & 0 & 0.1 \\ B & 0 & 0 & 0.4 \\ C & 0 & 1 & 0.3 \\ D & 0 & 0 & 0.2 \end{bmatrix} \]

\[ A = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.4 \\ 0.2 \end{bmatrix} \]

\[ B = \begin{bmatrix} 0.15 \\ 0.15 \\ 0.25 \\ 0.25 \end{bmatrix} \]

\[ \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ \mathbf{A} = (\mathbf{I} - \mathbf{B})^{-1} \]
Example: In terms of long term behavior, what percent of the time will you end up in state A if you start off in State B? State C? State E?

\[
T = \begin{bmatrix}
A & B & C & D & E \\
A & 1 & 0 & 0.1 & 0.3 & 0.2 \\
B & 0 & 1 & 0.3 & 0.2 & 0.1 \\
C & 0 & 0 & 0.2 & 0.1 & 0.3 \\
D & 0 & 0 & 0.2 & 0.2 & 0.1 \\
E & 0 & 0 & 0.2 & 0.2 & 0.3 \\
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
\cdot1 & \cdot3 & \cdot2 \\
\cdot3 & \cdot2 & \cdot1 \\
\cdot2 & \cdot1 & \cdot3 \\
\cdot2 & \cdot2 & \cdot1 \\
\cdot2 & \cdot2 & \cdot3 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
\cdot2 & \cdot1 & \cdot3 \\
\cdot2 & \cdot2 & \cdot3 \\
\cdot2 & \cdot2 & \cdot3 \\
\end{bmatrix}
\]

\[
A * (I - A)^{-1}
\]

\[
L = \begin{bmatrix}
A & B & C & D & E \\
A & 1 & 0 & 0.3984 & 0.5590 & 0.5365 \\
B & 0 & 1 & 0.6011 & 0.4410 & 0.4635 \\
C & 0 & 0 & 0 & 0 & 0 \\
D & 0 & 0 & 0 & 0 & 0 \\
E & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\text{Start} \quad \%\text{end up in A} \\
B \quad 0 \\
C \quad 39.89\% \\
E \quad 53.65\%
\]
Theorem: (Part 2) The computation $F = (I - B)^{-1}$ is called the fundamental matrix.

a) the sum of the entries in a column is the expected number of time it will take to enter an absorbing state if you start in that state(column label).

b) the individual entries in a column are the expected number of times of being in a non absorbing state(row label) if you start in particular state(column label).
Example: Find the fundamental matrix.

\[
T = \begin{bmatrix}
A & C & B & D \\
1 & 0 & 0.1 & 0.1 \\
0 & 1 & 0.2 & 0 \\
0 & 0 & 0.6 & 0.2 \\
0 & 0 & 0.1 & 0.7 \\
\end{bmatrix}
\]

\[
B = D \begin{bmatrix}
0.6 & 0.2 \\
\end{bmatrix}
\]

\[
I = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

\[
F = (I - B)^{-1} = B \begin{bmatrix}
3 & 2 \\
0 & 4 \\
\end{bmatrix}
\]

If you start in \( D \) it will take (on avg) 6 times to be absorbed.

2 times in state B + 4 times in state D.
Example: Find the fundamental matrix.

\[ T = \begin{bmatrix}
A & B & C & D & E \\
1 & 0 & 0 & 0.1 & 0.2 \\
0 & 1 & 0 & 0.2 & 0.3 \\
0 & 0 & 1 & 0.3 & 0.1 \\
0 & 0 & 0 & 0.3 & 0.2 \\
0 & 0 & 0 & 0.1 & 0.2 \\
\end{bmatrix} \]

\[ B = \begin{bmatrix}
0.3 \\
0.1 \\
0.2 \\
\end{bmatrix} \]

\[ D \quad E \]

\[ F = (I - B)^{-1} = \begin{bmatrix}
1.4815 & 0.3704 \\
0.1812 & 1.2763 \\
\end{bmatrix} \]
Example: Heather and Blake play a card game in which they take turns drawing a card from a standard deck of cards. Heather can win the game if she draws a heart and Blake can win the game if he draws a black card. When a player doesn’t win on their turn, their card is returned to the deck, the deck is reshuffled, and it becomes the other players turn. The game has four states: Heather wins (HW), Blake wins (BW), Heather’s turn (HT), and Blake’s turn (BT).

A) Draw the transition diagram for this Markov process.

![Transition Diagram]

\[
T = \begin{pmatrix}
1 & 0 & 0.25 & 0 \\
0 & 1 & 0 & 0.5 \\
0 & 0 & 0 & 0.75 \\
0 & 0 & 0.5 & 0
\end{pmatrix}
\]
B) Find the limiting matrix.

\[
L = \frac{H_w}{B_w} \begin{bmatrix} \bar{I} & \begin{bmatrix} 1 & .2 \\ .6 & .8 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} \end{bmatrix}
\]

C) What is the probability that Heather wins if she goes first? \( \frac{4}{4} \)

D) What is the expected number of turns if Heather goes first?

\[
F = \begin{bmatrix} \bar{H} & \bar{B} \\ \begin{bmatrix} 1.4 & .8 \\ 1.2 & .6 \end{bmatrix} \end{bmatrix}
\]

1.6 + 1.2 = 2.8 turns.

E) If each player takes 2 minutes to select their card, what is the expected length of time that the game will take if Heather goes first?

\[2.8 \text{ (2 min)} = 5.6 \text{ min.}\]

F) If Heather takes 3 minutes to select her card and Blake takes 1 minute to select his card, what is the expected length of time that the game will take if Blake goes first?

\[.8(3) + .6(1) = 4 \text{ min.}\]