Section M.2: Regular Markov Processes

Definition: A transition matrix (stochastic matrix) is said to be regular if some power of $T$ has all positive entries. This means that the Markov chain represented by $T$ is called a regular Markov chain.

Example: Are the transition matrices regular?

A) $T = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$

B) $T = \begin{bmatrix} 0.3 & 0 & 0.6 \\ 0 & 0 & 0.4 \\ 0.7 & 1 & 0 \end{bmatrix}$

C) $T = \begin{bmatrix} 0 & 0.2 & 1 \\ 0 & 0.3 & 0 \\ 1 & 0.5 & 0 \end{bmatrix}$

D) $T = \begin{bmatrix} 1 & 0.2 & 0.1 & 0.1 \\ 0 & 0.4 & 0.1 & 0 \\ 0 & 0.2 & 0.2 & 0.5 \\ 0 & 0.2 & 0.5 & 0.4 \end{bmatrix}$
**Definition:** A Markov process that does not have a change in the next distribution state has reached a steady state or equilibrium state. The steady state is denoted by $X_L$.

A Markov process that has a regular transition matrix will have a steady state.

Example: From the car/bus problem from M-1. Find the steady state distribution.

$$T = \begin{bmatrix} \text{car} & \text{bus} \\ \text{car} & 0.7 & 0.1 \\ \text{bus} & 0.3 & 0.9 \end{bmatrix}$$

$$X_L = \begin{bmatrix} A \\ B \end{bmatrix}$$

$$TX_L = X_L$$

$$\begin{bmatrix} .7 & .1 \\ .3 & .9 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$$

$$0.7A + 0.1B = A$$
$$0.3A + 0.9B = B$$

$$-0.3A + 0.1B = 0$$
$$0.3A - 0.1B = 0$$
$$A + B = 1$$

Use ref

$$\begin{bmatrix} -0.3 & 0.1 & 0 \\ 0.3 & -0.1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0.25 \\ 0 & 1 & 0.75 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = 0.25$$
$$B = 0.75$$

**Steady state:** $X_L = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}$
Example: Find the steady state for the pizza problem from M-1.

\[
T = \begin{bmatrix}
A & B & C \\
0.5 & 0.4 & 0.25 \\
0.3 & 0.2 & 0.5 \\
0.2 & 0.4 & 0.25 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} = 
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}
\]

\[
0.5x + 0.4y + 0.25z = x \\
0.3x + 0.2y + 0.5z = y \\
0.2x + 0.4y + 0.25z = z
\]

Use rref and get

\[
\chi_L = \begin{bmatrix}
0.3980 \\
0.3234 \\
0.2786 \\
\end{bmatrix} = \begin{bmatrix}
80/201 \\
65/201 \\
56/201 \\
\end{bmatrix}
\]
Definition: The limiting matrix or the stable matrix is defined as \( L = \lim_{n \to \infty} T^n \).

\[
T = \begin{bmatrix}
0.7 & 0.1 \\
0.3 & 0.4
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
0.25 & 0.25 \\
0.75 & 0.75
\end{bmatrix}
\]