Section G.1: Decision Making

Definition: A 'game' is where two (or more) opponents are involved in a competitive situation, with each opponent seeking to maximize their 'gains' or equivalently to minimize their 'losses.'

Definition: A zero-sum game is a game where payoff for one party results in an equal loss to the other party. A constant-sum game is a game where the payoff to both parties are related to a constant. For example, market share for two competing businesses.

Definition: The payoff matrix is a matrix that relates the strategies of the players and the results of those strategies with respect to a particular player. We will let the entries represent the Row Player’s results.

\[
\begin{array}{cccc}
\text{Row's strategy} & & \text{Column's strategy} \\
\text{R-1} & a_{11} & a_{12} & \cdots & a_{1n} \\
\text{R-2} & a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\text{R-m} & a_{m1} & a_{m2} & \cdots & a_{mn} \\
\end{array}
\]
Example: Rich and Chuck play a coin matching game where each player selects the side of a coin (without the other’s knowledge) and discloses their choice at the same time. They have agreed to the following results: If both coins are heads, then Chuck pays Rich $5. If both coins are tails, then Chuck pays Rich $1. If Chuck shows tails and Rich heads, then Rich pays Chuck $8. If Chuck shows heads and Rich shows tails, then Chuck pays Rich $2. Give the payoff matrix from Rich’s point of view.
Example: Rick and Carl are playing: Rock, Paper, Scissors. The winner of each round is paid $1 by the loser. No money is exchanged on ties. Remember that paper beats rock; rock beats scissors; and scissors beats paper. Give Rick's payoff matrix.

\[ \begin{bmatrix}
  0 & -1 & 1 \\
  1 & 0 & -1 \\
  -1 & 1 & 0
\end{bmatrix} \]
Example: John has two cards: a red 4 and a black 8. David has four cards: a red 8, a red 9, a black 4 and a black 10. The game that they are going to play is to each pick a card and play it at the same time. If the cards are the same color, then John receives the difference of the two numbers. If the cards are different colors, David receives the minimum of the two numbers. **Give the payoff matrix from John’s point of view.**

<table>
<thead>
<tr>
<th></th>
<th>( \text{R-8} )</th>
<th>( \text{R-9} )</th>
<th>( \text{B-4} )</th>
<th>( \text{B-10} )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>R-4</strong></td>
<td>4</td>
<td>5</td>
<td>-4</td>
<td>-4</td>
</tr>
<tr>
<td><strong>B-8</strong></td>
<td>-8</td>
<td>-8</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

**David**
Determining Strategies for a Game

Definition: A **pure strategy** is when the player uses the same strategy (same row choice/column choice) every time. Strategies that involve varied choices are called **mixed strategies**.

Example: The payoff matrix for a zero-sum game is given below. The numbers represent the number of dollars won(loss if negative) by the row player.

A) Discuss the best strategy for row player.

\[
\begin{bmatrix}
15 & -3 & -4 & 2 \\
12 & 9 & 6 & 8 \\
-5 & -2 & 3 & 16
\end{bmatrix}
\]

15 \[\text{min for the row}\]

-4 \[\text{best of the mins}\]

6 \[\text{max}\]

-5 \[\text{max}\]

**Best pure strategy for the row player is Row 2.**

---

**Pure Strategy for the Row player: (Maximin Strategy)**

1) For each row of the payoff matrix determine the \[\text{minimum}\] element.

2) Choose the row that has the \[\text{maximum}\] element from step 1.
B) Discuss the best strategy for column player.

\[
\begin{pmatrix}
15 & -3 & -4 & 2 \\
12 & 9 & 6 & 8 \\
-5 & -2 & 3 & 16
\end{pmatrix}
\]

Best Pure Strategy
for the column
player is C3

\[\text{Smallest of Row #s}\]

---

**Pure Strategy for the Column player: (Minimax Strategy)**

1) For each column of the payoff matrix determine the **max** element.

2) Choose the column that has the **min** element from step 1.
Example: Find the Pure strategy for both the row and the column player.

\[
\begin{bmatrix}
-3 & -2 & 4 \\
-2 & 0 & 3 \\
5 & -1 & 1
\end{bmatrix}
\]

\[
\begin{align*}
\min & \quad \text{min} \\
-3 & \quad -2 \\
-2 & \quad -2 \\
-1 & \quad -1 \\
\end{align*}
\]

\text{Max } 5 \text{ so } y

\text{Best Pure strategy for col player is Col 2}

\text{Best Pure strategy for row player is Row 3.}
**Definition:** If there is an entry in the payoff matrix that is simultaneously the smallest entry in its row and the largest entry in its columns, this entry is called a **saddle point**.

**Definition:** A game is called strictly determined if the payoff matrix has a saddle point.

**Definition:** The saddle point is also called the **value of the game**. If the value of the game is positive (negative), then the game favors the row player (column player).

If the value of the game is zero, then the game is fair.

The **location(s)** of the saddle point is the optimal strategy for the row player and column player.
Find the saddle point of this payoff matrix.

\[
\begin{bmatrix}
  15 & -3 & -4 & 2 \\
  12 & 9 & 6 & 8 \\
 -5 & -2 & 3 & 16
\end{bmatrix}
\]

\[
\begin{align*}
\text{min} & : \\
& -4 \\
& 6 \\
& -5
\end{align*}
\]

\[
\begin{align*}
\text{max} & : \\
& 15 \\
& 9 \\
& 4 \\
& 16
\end{align*}
\]

Saddle point in R2 C3
Example: Determine if the two-person game is strictly determined. If the game is strictly determined,
i) find the saddle point(s) of the game.
ii) find the optimal strategy of the game.
iii) find the value of the game and who the game favors.

\[
\begin{bmatrix}
-1 & 2 & 4 \\
2 & 3 & 5 \\
0 & 1 & -3 \\
-2 & 4 & -2 \\
\end{bmatrix}
\]
B) \[
\begin{bmatrix}
2 & 1 & -2 & 4 \\
1 & -3 & -5 & 1 \\
3 & 2 & 4 & -2
\end{bmatrix}
\]
\[
\begin{align*}
\text{max} & \quad 3 & 2 & 4 & 4 & \quad \text{min} & \quad -2 & -5 & -2 \\
\text{no saddle point}
\end{align*}
\]

C) \[
\begin{bmatrix}
3 & 5 & 3 \\
0 & 4 & 0 \\
-2 & -3 & 2
\end{bmatrix}
\]
\[
\begin{align*}
\text{max} & \quad 3 & 5 & 3 & \quad \text{min} & \quad 3 & 0 & -3 \\
\text{saddle point} & \quad R1 \text{ C1 and R1 C3} & \quad \text{Value of game} = 3 & \quad \text{favors Row player}
\end{align*}
\]

\[
\begin{array}{c}
\text{Row player} \\
\hline
R1
\end{array} \quad \begin{array}{c}
\text{Col. player} \\
\hline
C1 \text{ C3}
\end{array}
\]