$V = \ell \cdot w \cdot h$

$= (\text{Area of base}) \cdot h$

$V = \pi r^2 h$
Section 7.2: Volume

Let $S$ be a solid that lies between the planes $P_a$ and $P_b$. Assume that cross sections of the solid is given by $A$ and are perpendicular to the x-axis.

\[
V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i) \Delta x_i = \int_{a}^{b} A(x) \, dx
\]
1. The solid, S, has a base that is a circular disk with radius 2. Find the volume of the solid if parallel cross sections taken perpendicular to the base are squares.

\[ x^2 + y^2 = 2^2 = 4 \]

Area of cross section:
\[ A = \pi r^2 = \pi y^2 \]

\[ \int_{-2}^{2} 4y^2 \, dx = \int_{-2}^{2} 4(4-x^2) \, dx \]

\[ = \int_{-2}^{2} 16 - 4x^2 \, dx = \frac{128}{3} \]
2. The solid, S, has a base that is bounded by the equations: \( y = x^2 \) and \( y = 4 \). Find the volume of the solid if parallel cross sections are equilateral triangles that are perpendicular to the y-axis.

\[
\text{Area of cross section} = \frac{1}{2} b h = \frac{1}{2} (2x) \cdot h = \frac{1}{2} (2x) \cdot x\sqrt{3} \\
A = x^2 \sqrt{3}
\]

\[
\int_0^4 x^2 \sqrt{3} \, dy = \int_0^4 y \sqrt{3} \, dy = \cdots = 8\sqrt{3}
\]
Now let's consider rotating a region bounded between the x-axis and the function $f(x)$ from $x = a$ to $x = b$ around the x-axis.

\[
A = \pi r^2 = \pi y^2
\]

\[
V = \int_a^b \pi (f(x))^2 \, dx
\]
3. Find the volume of the solid obtained by rotating the region bounded by the following around the $x$-axis.

$y = x^2 + 1$

$x$-axis

$x = -1$

$x = 2$

$r = y = x^2 + 1$

\[
V = \pi \int_{-1}^{2} \pi (x^2 + 1)^2 \, dx = \pi \int_{-1}^{2} x^4 + 2x^2 + 1 \, dx
\]

\[
= \pi \left[ \left( \frac{x^5}{5} + \frac{2x^3}{3} + x \right) \right]_{-1}^{2}
\]

\[
= \pi \left[ \left( \frac{2^5}{5} + \frac{2(2)^3}{3} + 2 \right) - \left( \frac{(-1)^5}{5} + \frac{2(-1)^3}{3} + (-1) \right) \right]
\]

\[
= \ldots = 15.6 \pi
\]
4. Find the volume of the solid obtained by rotating the region bounded by the given curves around the \(y\)-axis.

\[x = 4y - y^2 = y(4-y)\]
\[x = 0\]

\[r = x = 4y - y^2\]

\[V = \int_0^4 \pi (4y - y^2)^2 \, dy = \ldots = \frac{512}{15} \pi\]
5. Find the volume of the solid obtained by rotating the region bounded by the given curves around the $y$-axis.

\[ y = x^2 + 1 \]
\[ y = 0 \]
\[ x = 0 \]
\[ x = 2 \]

Total volume of cylinder

\[ V = \pi (2)^2 \cdot 5 = 20\pi \]

Volume of pink region

\[ V = \int_{1}^{5} \pi x^2 \, dy = \int_{1}^{5} \pi (y^2 - 1) \, dy = 8\pi \]

Answer

\[ \text{Vol.} = 20\pi - 8\pi = 12\pi \]
5. Find the volume of the solid obtained by rotating the region bounded by the given curves around the y-axis.

\[ y = x^2 + 1 \]
\[ y = 0 \]
\[ x = 0 \]
\[ x = 2 \]

\[
V = \pi \int r^2 h = \pi \int_0^2 (x)^2 \cdot 1 = 4\pi
\]

\[
\begin{align*}
\frac{\text{part 1}}{r_\text{o}} &= 2 \\
\frac{\text{part 2}}{r_\text{e}} &= x
\end{align*}
\]

\[
V = \int_1^5 \pi (2)^2 - \pi (x)^2 \, dy
\]

\[
= \int_1^5 \pi (y - x^2) \, dy
\]

\[
= \int_1^5 \pi (y - (y-1)) \, dy
\]

\[
= 8\pi
\]

Answer = \text{part 1} + \text{part 2} = 4\pi + 8\pi = 12\pi
Now let's consider rotating a region bounded between the function $f(x)$ and $g(x)$ from $x = a$ to $x = b$ around the x-axis.

\[ A = \pi r_o^2 - \pi r_a^2 \]

\[ V = \int_a^b \pi (r_2^2 - r_1^2) \, dx \]
6. Set up the integral(s) that would give the volume of the solid obtained by rotating the region bounded by the given curves around $x$-axis.

$y = x^2 + 2$
$2y - x = 2$
$x = 0$
$x = 1$

\[ y = \frac{1}{2}x + 1 \]

\[ \Gamma_0 = y = x^2 + 2 \]
\[ \Gamma_1 = y = \frac{1}{2}x + 1 \]

\[ V = \int_{0}^{1} \pi \left[ (x^2 + 2)^2 - \left( \frac{1}{2}x + 1 \right)^2 \right] \, dx \]

\[ = \cdots = \frac{\pi}{60} \]
7. Set up the integral(s) that would give the volume of the solid obtained by rotating the region bounded by the given curves around \( y = 3 \)

\( y = x^2 + 2 \)

\( 2y - x = 2 \)

\( x = 0 \)

\( x = 1 \)

\[ y = \frac{1}{2} x + 1 \]

\[ y + r_2 = 3 \]

\[ r_2 = 3 - y = 3 - \left( \frac{1}{2} x + 1 \right) = 2 - \frac{1}{2} x \]

\[ r_i = 3 - y = 3 - (x^2 + 2) = 1 - x^2 \]

\[ V = \int_0^1 \pi \left[ \left( 2 - \frac{1}{2} x \right)^2 - (1 - x^2)^2 \right] \, dx \]

\[ = \cdots = \frac{51}{20} \pi \]
8. Set up the integral(s) that the volume of the solid obtained by rotating the region bounded by the given curves around $x = 4$.

\[
y = x^3 \\
y = 3x + 2 \\
x = 0
\]

\[
V = \pi \left[ \int_0^2 \left( y^2 - \left(4 - \frac{3\sqrt[3]{y}}{2}\right)^2 \right) \, dy + \int_2^8 \left( y - \frac{y^2}{3} \right)^2 - \left(4 - \frac{3\sqrt[3]{y}}{2}\right)^2 \, dy \right]
\]

\[
= \cdots = \frac{184\pi}{5}
\]