Spring 2008 Math 151

Week in Review # 6
sections: 3.10-3.12
courtesy: Joe Kahlig

1. The length of a rectangle is increasing at a rate of 2 feet per second, while the width is increasing at a rate of 1 foot per second. When the length is 5 feet and the width is 3 feet, how fast is the area increasing?

\[
\frac{dA}{dt} = \frac{d}{dt}(lw) = w \frac{dl}{dt} + l \frac{dw}{dt}
\]

\[
= 3(2) + 5(1) = 11 \text{ ft}^2/\text{sec}
\]

2. A point moves around the ellipse \(4x^2 + 9y^2 = 75\).
When the point is at \((\sqrt{3}, \sqrt{7})\), its \(x\) coordinate is increasing at a rate of 10 units per second. What is the rate of change of the \(y\) coordinate at that instant?

\[
8x \frac{dx}{dt} + 18y \frac{dy}{dt} = 0
\]

\[
8(\sqrt{3})(10) = -18(\sqrt{7}) \frac{dy}{dt}
\]

\[
- \frac{80\sqrt{3}}{18\sqrt{7}} = \frac{dy}{dt}
\]

\[
\frac{dy}{dt} = -2.90957 \text{ units/sec}
\]
3. You want to fly a kite so that it is 100 ft above the ground and moving horizontally at a speed of 8 ft/sec. At what rate should the string be released when 260 feet of string has been let out. Assume that there is no slack in the string and ground is level.

\[ \frac{dx}{dt} = \frac{dy}{dt} \]

\[ x^2 + 100^2 = y^2 \]

\[ 2x \frac{dx}{dt} = 2y \frac{dy}{dt} \]

\[ \frac{2y}{260} = \frac{dy}{dt} \rightarrow \frac{dy}{dt} = \frac{7.384}{6} \text{ ft/sec} \]

4. A ladder 15 feet long rests against a vertical wall. If the top of the ladder slides down the wall at a speed of 1.5 feet per second, at what rate of change is the angle between the bottom of the ladder and the ground changing when the angle is \( \frac{\pi}{4} \) radians? Assume that the ground is level.

\[ \sin \theta = \frac{15}{y} \]

\[ \cos \theta \frac{dy}{dt} = \frac{15}{10} \frac{dy}{dt} \]

\[ \cos \left( \frac{\pi}{4} \right) \frac{d\theta}{dt} = \frac{1}{15} (-1.5) \]

\[ \frac{1}{\sin \theta} \frac{d\theta}{dt} = -1 \]

\[ \frac{d\theta}{dt} = -1.5 \text{ rad/sec} \]
5. At noon ship A leaves a port traveling north at 40 km/hr. Ship B leaves the same port 1 hour later traveling east at 25 km/hr. How fast is the distance between the ships changing at 4pm?

\[
\frac{dx}{dt} = 40 \text{ km/hr} \\
\frac{dy}{dt} = 25 \text{ km/hr}
\]

At 4pm:
\[
x = 120 \text{ km} \\
y = 75 \text{ km}
\]

\[
(x + y)^2 + y^2 = z^2
\]

\[
2(x + y) \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}
\]

\[
(x + y) \frac{dx}{dt} + y \frac{dy}{dt} = 2 \frac{dz}{dt}
\]

\[
160(40) + 75(25) = \frac{dz}{dt}
\]

\[
\frac{dz}{dt} = 4.829 \text{ km/hr}
\]

6. A water tank has the shape of an inverted right circular cone of altitude 18 ft and a base radius of 6 ft. If water is being pumped into the tank at a rate of 10 gal/min (\(\approx 1.337 \text{ ft}^3/\text{min}\)), find the rate at which the water level is rising when the water is 5 ft deep.

\[
V = \frac{1}{3} \pi r^2 h
\]

\[
V = \frac{1}{3} \pi \left(\frac{1}{2} h\right)^2 h
\]

\[
V = \frac{1}{27} \pi h^3
\]

\[
\frac{dV}{dt} = \frac{3\pi}{27} h^2 \frac{dh}{dt}
\]

\[
\frac{dV}{dt} = \frac{\pi}{9} (5)^2 \frac{dh}{dt}
\]

\[
\frac{(133.7)}{25\pi} = \frac{dh}{dt}
\]

\[
\frac{dh}{dt} = 0.153 \text{ ft/min}
\]
7. Given \( y = 4 - x^2 \)

(a) Find \( \Delta y \) if \( x \) changes from \( x = 1 \) to \( x = 1.5 \)

\[
\begin{align*}
X = 1 & \quad y = 3 \\
X = 1.5 & \quad y = 4 - 2.25 = 1.75
\end{align*}
\]

\( \Delta y = 1.75 - 3 = -1.25 \)

(b) Find \( dy \) for \( x = 1 \) and \( dx = 0.5 \)

\[
dy = -2x \, dx
\]

\[
dy = -2(1)(0.5) = -1
\]

8. Use differentials to approximate:

(a) \( \sqrt{16.4} \)

\[
\begin{align*}
X = 16 & \quad y = \sqrt{x} \\
Ax = dx & \quad dy = \frac{1}{2} \sqrt{x} \, dx
\end{align*}
\]

\[
f(x + Ax) = f(x) + dy = 4 + \frac{1}{8} = 4.125
\]

(b) \( \cos 28^\circ \)

\[
\begin{align*}
X = \frac{\pi}{6} & \quad y = \cos x \\
Ax = -\frac{2\pi}{180} & \quad dy = -\sin (x) \, dx
\end{align*}
\]

\[
\begin{align*}
\cos (28^\circ) & \approx \cos \left( \frac{\pi}{6} \right) + dy \\
& = \frac{\sqrt{3}}{2} + \frac{\pi}{180}
\end{align*}
\]
9. Find the linear approximation for \( y = \sqrt{1 + x} \) at \( a = 0 \) and use it to approximate \( \sqrt{.95} \) and \( \sqrt{1.2} \):

\[
L(x) = f(a) + f'(a)(x-a)
\]

\[
f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}
\]

\[
f'(0) = \frac{1}{2}(1)^{-\frac{1}{2}} = \frac{\sqrt{2}}{2}
\]

\[
L(x) = 1 + \frac{1}{2}(x-0) = 1 + \frac{1}{2}x
\]

\[
\sqrt{.95} = \sqrt{1 + -.05} \rightarrow L(-.05) = 1 + \frac{1}{4}(-.05) = 1.9875
\]

\[
\sqrt{1.2} = \sqrt{1 + .2} \rightarrow L(.2) = 1 + \frac{1}{4}(.2) = 1.05
\]

10. Find the linear approximation for \( y = \frac{1}{x} \) at \( x = \frac{1}{4} \):

\[
y' = -\frac{1}{x^2}
\]

\[
y'(\frac{1}{4}) = -\frac{1}{\left(\frac{1}{4}\right)^2} = -16 = m_{lin}
\]

\[
y(\frac{1}{4}) = \frac{1}{4} = y
\]

Tangent line formula:

\[
y - y = m_{lin}(x - \frac{1}{4})
\]

\[
y = y - 16\left(x - \frac{1}{4}\right)
\]

Linearization formula:

\[
L(x) = y(\frac{1}{4}) - y'(\frac{1}{4})(x - \frac{1}{4}) = y - 16\left(x - \frac{1}{4}\right)
\]

\[
y = y - 16x + y
\]

\[
L(x) = 8 - 16x
\]
11. Find the quadratic approximation for \( y = \frac{1}{x^2} \) at \( a = 2 \).

\[
Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2
\]

\[
f(x) = x^{-2} \quad f(2) = \frac{1}{4}
\]
\[
f'(x) = -2x^{-3} \quad f'(2) = -\frac{2}{8} = -\frac{1}{4}
\]
\[
f''(x) = 6x^{-4} \quad f''(2) = \frac{6}{16} = \frac{3}{8}
\]

\[
Q(x) = \frac{1}{4} - \frac{1}{4}(x-2)^2 + \frac{3}{8} (x-2)^2
\]

\[
Q(x) = \frac{1}{4} + \frac{1}{4}(x-2)^2 + \frac{3}{2} (x-2)^2
\]

12. The radius of a circular disk is given as 40 cm with a maximum error in measurement of 0.3 cm. Use differentials to estimate the maximum error in the calculated area of the disk. What is the relative error?

\[
r = 40 \text{ cm} \\
dr = \pm 0.3 \text{ cm}
\]

\[
A = \pi r^2
\]

\[
dA = 2\pi r \, dr
\]

\[
= 2\pi (40)(\pm 0.3)
\]

\[
= \pm 24\pi \text{ cm}^2
\]

\[
\text{Relative Error} = \frac{dA}{A} = \pm \frac{24\pi}{1600\pi}
\]

\[
= \pm \frac{3}{200} = \pm .015
\]
13. Given \( f(x) = x^3 + x^2 - 2 \), use Newton’s Method with \( x_1 = 2 \) to find the third approximation to the root of the given equation.

\[
\begin{align*}
X_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
X_{n+1} &= x_n - \frac{x_n^3 + x_n^2 - 2}{3x_n^2 + 2x_n}
\end{align*}
\]

\( x_1 = 2 \)  
\( x_2 = 1.375 \)  
\( x_3 = 1.0793135 \)
14. Use Newtons method to approximate $\sqrt[3]{20}$ to 6 decimal places.

\[
x^3 - 20 = 0
\]
\[
f(x) = x^3 - 20
\]
\[
x_{n+1} = x_n - \frac{x_n^3 - 20}{3x_n^2}
\]

\[
x_1 = 3
\]
\[
x_2 = 2.7407407
\]
\[
x_3 = 2.7146696
\]
\[
x_4 = 2.71441764
\]
\[
x_5 = 2.714417617
\]

15. Use Newtons Method to approximate the solution of $x^4 - 8x^2 - 5x = -18$ on the interval $[1, 2]$ to 6 decimal places.

\[
x^4 - 8x^2 - 5x + 18 = 0
\]
\[
f(x) = x^4 - 8x^2 - 5x + 18
\]
\[
x_{n+1} = x_n - \frac{x_n^4 - 8x_n^2 - 5x_n + 18}{4x_n^3 - 16x_n - 5}
\]

\[
x_1 = 1
\]
\[
x_2 = 1.352941176
\]
\[
x_3 = 1.349491108
\]
\[
x_4 = 1.34949328
\]
\[
x_5 = 1.34949321
\]